Lecture 19

Thursday, June 9, 2022 9:33 PM

* Prayer

* Spiritual thought

Methods to compute a line integral $\int_C F \cdot dr$:

- Use parametrization:

$$\int_C F \cdot dr = \int_a^b F(r(t)) \cdot r'(t)\, dt$$

- Use fundamental of Calculus: (only applicable if F is conservative)

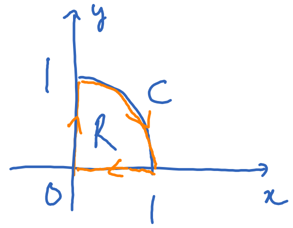
$$\int_C F \cdot dr = f(B) - f(A)$$

- Use Green's theorem! (only applicable for loops)

$$\int_C F \cdot dr = \pm \iint_D \underbrace{(Q_x - P_y)}_{\text{curl } F} dA$$

Plus sign is chosen if C is positively oriented. Otherwise, minus sign is chosen.

Ex



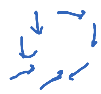
$$\iint_C \underbrace{xy}_{P}\, dx + \underbrace{y^2}_{Q}\, dy = -\iint_R (Q_x - P_y)\, dA$$

$$= -\iint_R -x\, dA = \iint_R x\, dA = \int_0^1 \int_0^{\pi/2} r^2 \cos\theta\, d\theta\, dr$$

Recall: $\text{curl } F = Q_x - P_y$ is the circulation density at (x,y).

If $\text{curl } F(x,y) > 0$ then the flow F tends to circulate in a positive orientation around the point (x,y). If $\text{curl } F(x,y) < 0$ then F tends to circulate in a negative orientation.

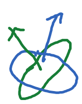
$\text{curl } F > 0$



$\text{curl } F < 0$

How about 3D flows?

Circulation density should depends on the "direction" the curve is facing.



$\text{curl } F = w$

$w \cdot n$ = circulation density in the direction n.

Formula: $F = (P, Q, R)$

$$\text{curl } F = \nabla \times F = \begin{matrix} \partial_x & \partial_y & \partial_z \\ P & Q & R \end{matrix} = (R_y - Q_z,\ P_z - R_x,\ Q_x - P_y)$$

Ex $F = (xy, yz, zx)$

$\text{curl } F = ?$

* Divergence of a vector field measures the tendency a flow emanates from a point.

$\operatorname{div} F = \nabla \cdot F$

In 2D: $\operatorname{div} F = (\partial_x, \partial_y) \cdot (P, Q) = P_x + Q_y$

In 3D: $\operatorname{div} F = (\partial_x, \partial_y, \partial_z) \cdot (P, Q, R) = P_x + Q_y + R_z$

Divergence = flux density

Curl = circulation density.

Ex $F = (xy, y^2)$ at $(-1, 1)$

$\operatorname{div} F = x + 2y$

$\operatorname{div} F(-1,1) = -1 + 2 = 1$

Surface integral

$\iint_S f \, \underbrace{dS}_{\text{small cell on a surface}}$, $\iint_S \vec{F} \cdot \underbrace{d\vec{S}}_{\text{small vector normal to the surface}}$

Surface parametrization:

$r(u,v) = (x(u,v), y(u,v), z(u,v))$, where $(u,v) \in R$.

flat region

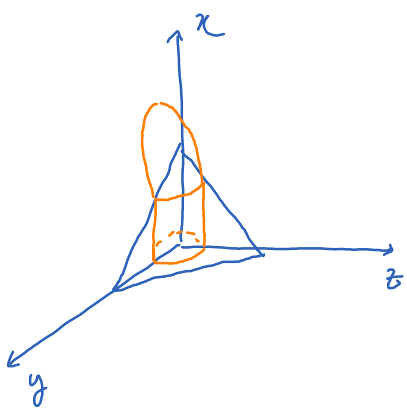
Ex The hemisphere $x^2 + y^2 + z^2 = 4$, $z > 0$.

$$\begin{cases} x = 4\sin\phi\cos\theta \\ y = 4\sin\phi\sin\theta \\ z = 4\cos\phi \end{cases} \qquad \boxed{\begin{array}{l} 0 \le \theta \le 2\pi \\ 0 \le \phi \le \frac{\pi}{2} \end{array}}\ R$$

Ex The surface (wall) built on the parabola $y = x^2$ (on the xy-plane) from $(0,0)$ to $(2,4)$ and under the paraboloid $z = x^2 + y^2$.

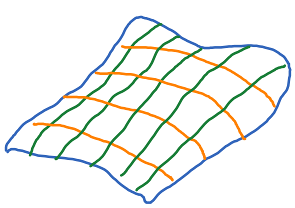
$$\begin{cases} x = t \\ y = t^2 \\ z = z \end{cases} \qquad \boxed{\begin{array}{l} 0 \le t \le 2 \\ 0 \le z \le t^2 + t^4 \end{array}}\ R$$

Ex The surface on the cylinder $y^2 + z^2 = 1$ bounded by the plane $x = 0$ and $x + y + z = 3$.



$$\begin{cases} y = \cos t \\ z = \sin t \\ x = x \end{cases} \qquad \begin{array}{l} 0 \le t \le 2\pi \\ 0 \le x \le 3 - y - z \\ \quad = 3 - \cos t - \sin t \end{array}$$

Tangent plane of a surface:



$S: r(u,v)$

orange line : u fixed, v moves

green line : u moves, v fixed

Tangent vector of the orange curve: $\frac{\partial r}{\partial v} = (x_v, y_v, z_v)$

Tangent vector of the green curve: $\frac{\partial r}{\partial u} = (x_u, y_u, z_u)$

$\rightsquigarrow$ normal vector of tangent plane is $r_u \times r_v$.

Ex The surface $r(u,v) = (u^2+v^2, u-v, u+v)$ passes through $(2,2,0)$.

Find the tangent plane at this point.

$$u = 1, \quad v = -1$$

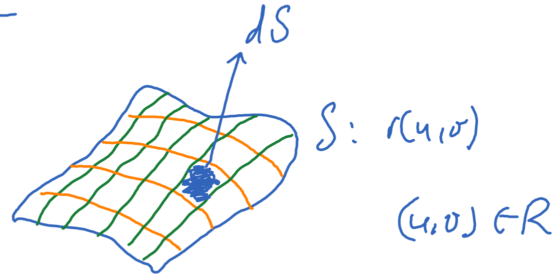
$$\left.\begin{aligned} r_u &= (2u, 1, 1) = (2,1,1) \\ r_v &= (2v, -1, 1) = (-2,-1,1) \end{aligned}\right\} r_u \times r_v = (2, -4, 0)$$

$\rightsquigarrow$ Eq. of tangent plane is $2(x-2) - 4(y-2) + 0(z-0) = 0$

$$\rightsquigarrow \quad x - 2y + 2 = 0$$

Surface integral of a scalar function

$$\iint_S f \, dS$$



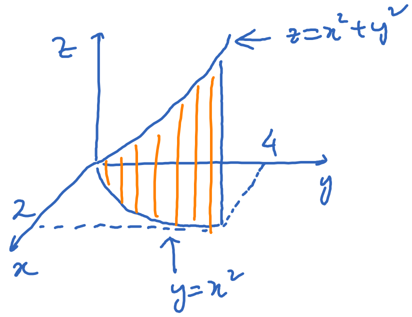
$$dS = |r_u du \times r_v dv|$$

$$= |r_u \times r_v| \underbrace{du dv}_{dA}$$

Therefore,

$$\iint_S f\,dS = \iint_R f(x(u,v), y(u,v), z(u,v))\,|r_u \times r_v|\,dA$$

Ex:



$f(x,y,z) = xz$: mass density per unit area

Total mass $= ?$

$$\begin{cases} x = t \\ y = t^2 \\ z = z \end{cases} \longrightarrow r(t,z) = (t, t^2, z) \leadsto \begin{cases} r_t = (1, 2t, 0) \\ r_z = (0,0,1) \end{cases}$$

$$r_t \times r_z = (2t, 0, 0) \leadsto |r_t \times r_z| = 2t$$

$$\iint_S f\,dS = \iint_R tz\,dA = \int_0^2 \int_0^{t^2+t^4} tz\,dz\,dt = \cdots$$